Class-room notes on
Uniplanar Kinematics.

by

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CLASS-ROOM NOTES ON UNIPLANAR KINEMATICS

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Velocity in a Plane. A point moving in a plane is at any instant determined in position by a pair of co-ordinates x, y or r, θ, and its path may usually be defined by an equation, $y = f(x)$, or $r = \varphi(\theta)$. If ρ be the vector (directed magnitude) from the origin to the point x, y or r, θ, this path is more completely defined by the equations

$$\rho = x + iy, \quad y = f(x),$$

involving a complex function of the real variable x, or also, in terms of tensor and amplitude, by the equations

$$\rho = r \operatorname{cis} \theta, \quad r = \varphi(\theta).$$

The derivative of this vector ρ with respect to an equicrescent variable t, which may be taken to represent time, is its rate, and is defined as the *velocity* of the moving point whose position at any instant it determines. In terms of x and y this velocity is

$$\frac{d\rho}{dt} = \frac{dx}{dt} + i\,\frac{dy}{dt}$$

and has a component $\frac{dx}{dt}$ in the direction of the x-axis, a component $\frac{dy}{dt}$ in the direction of the y-axis, and the *speed* of the point in its path is the tensor of this velocity, or

$$\left|\frac{d\rho}{dt}\right| = \sqrt{\left\{\frac{dx}{dt}\right\}^2 + \left\{\frac{dy}{dt}\right\}^2}.$$

On the other hand, in terms of r and θ, the velocity is

$$\frac{d\rho}{dt} = \frac{dr}{dt} \operatorname{cis} \theta + ir \operatorname{cis} \theta \frac{d\theta}{dt},$$

and its components are

$\frac{dr}{dt} \operatorname{cis} \theta =$ velocity in the direction of the vector ρ

$ir \operatorname{cis} \theta \frac{d\theta}{dt} =$ velocity in a direction perpendicular to ρ,

and the speed of the point in its path is

$$\left| \frac{d\rho}{dt} \right| = \sqrt{\left\{ \frac{dr}{dt} \right\}^2 + \left\{ r \frac{d\theta}{dt} \right\}^2}.$$

The components of velocity in two directions perpendicular to each other as here determined are independent of each other in the sense that neither involves any motion whatever in the direction of the other. Obviously this is the case whatever be the mode of resolution into two rectangular directions.

Velocity is compounded in still another important way by introducing a second variable s, assumed to represent the length of the path of the moving point measured from a fixed point. In this form it is

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{ds}{dt} \frac{d\rho}{ds} \\ &= v\, \rho', \end{aligned}$$

where $v = \frac{ds}{dt}$ and $\rho' = \frac{d\rho}{ds}$, and since $ds^2 = dx^2 + dy^2$,

$$v = \sqrt{\left\{ \frac{dx}{dt} \right\}^2 + \left\{ \frac{dy}{dt} \right\}^2}.$$

That $| \rho' | = 1$ appears at once from the fact that

$$\frac{d\rho}{ds} = \frac{dx}{ds} + i \frac{dy}{ds}$$

and

$$\therefore \left| \frac{d\rho}{ds} \right|^2 = \left\{ \frac{dx}{ds} \right\}^2 + \left\{ \frac{dy}{ds} \right\}^2 = 1.$$

Hence v is the speed of the point along its path, as was asserted in the equation giving $\left|\frac{d\rho}{dt}\right|$ above, and ρ' is the velocity-direction, as is otherwise evident from the fact that $\underset{\Delta s = 0}{\text{limit}} \left\{\frac{\Delta\rho}{\Delta s}\right\}$ has the direction of a tangent to the path of the point at the instant considered.

Acceleration. The rate of change of velocity is called *acceleration.* Expressed as a derivative it is

$$\frac{d^2\rho}{dt^2} = \frac{d}{dt}\left\{\frac{ds}{dt}\frac{d\rho}{ds}\right\}$$

$$= \frac{d^2s}{dt^2}\frac{d\rho}{ds} + \frac{d^2\rho}{ds^2}\left\{\frac{ds}{dt}\right\}^2,$$

or if

$$\frac{ds}{dt} = v, \ \frac{d\rho}{ds} = \rho', \ \frac{d^2\rho}{ds^2} = \rho'',$$

it is

$$\frac{d^2\rho}{dt^2} = \frac{dv}{dt}\rho' + v^2\rho'',$$

and, like the analogous expressions for velocity, has two independent components $\frac{dv}{dt}\rho'$ and $v^2\rho''$, the former in the direction of the tangent and in magnitude equal to the rate of change of speed, the latter, as will be shown in the next paragraph, in a direction perpendicular to the tangent and equal in magnitude to the square of the speed multiplied by the curvature of the path at the point considered. The first is called tangenital, the second normal acceleration.

Radius of Curvature. Let the equations of a plane curve be

$$\rho = x + iy, \quad y = f(x),$$

and let differentiation with respect to the arc s, estimated from a fixed point, be denoted by accents, $\frac{d\rho}{ds} = \rho'$, $\frac{d^2x}{ds^2} = x''$, etc. Then

$$\rho' = x' + i\,y'$$
$$\rho'' = x'' = i\,y'',$$

and since $\quad |\,\rho\,| = 1$

$$\therefore \quad x'^2 + y'^2 = 1$$

and
$$x'x'' + y'y'' = 0.$$

Hence
$$\frac{\rho'}{\rho''} = \frac{x'x'' + y'y'' + i\,(x''y' - x'y'')}{x''^2 + y''^2}$$
$$= \frac{i\,(x''y' - x'y'')}{x''^2 + y''^2},$$

which shows that ρ'' is perpendicular to ρ', for i used as a multiplier turns any line in the plane through a right angle. And since $|\,\rho'\,| = 1$ and $|\,\rho''\,|^2 = x''^2 + y''^2$,

$$\therefore \quad |\,\rho''\,| = |\,x''y' - x'y''\,|\,.$$

But if φ be the angle between the x-axis and the tangent at the extremity of ρ

$$\underset{\Delta s = 0}{\text{limit}} \frac{\Delta x}{\Delta s} = x' = \cos\varphi, \text{ and } \underset{\Delta s = 0}{\text{limit}} \frac{\Delta y}{\Delta s} = y' = \sin\varphi,$$

or these expressions for $\sin\varphi$ and $\cos\varphi$ may be deduced from the equation $(dx/ds)^2 + (dy/ds)^2 = 1$ by assuming $dx/ds = \cos\varphi$, inferring therefrom $1 - (dx/ds)^2 = (dy/ds)^2 = \sin^2\varphi$ and identifying φ as the angle named. Then, by a second differentiation,

$$x'' = -\sin\varphi\,\frac{d\varphi}{ds},\;\; y'' = \cos\varphi\,\frac{d\varphi}{ds},$$

and thence, by cross multiplication with the previous equations,

$$x'y'' - x''y' = (\cos^2\varphi + \sin^2\varphi)\,\frac{d\varphi}{ds} = \frac{d\varphi}{ds}.$$

$\dfrac{d\varphi}{ds}$ is called the curvature of the curve at the point whose vector is ρ. Thus the assertion concerning the magnitude of the second, or normal component of acceleration, at the close of the last article, is verified.

The radius of curvature is the reciprocal of the curvature, or

$$R = \frac{ds}{d\varphi} = \pm\left|\,\frac{1}{\rho''}\,\right|.$$

Problems. Verify the following formulae for the determination of velocity and acceleration.

(1). If a point move with speed v in the curve $y = f(x)$, represented in Cartesian co-ordinates, prove that its velocity is

$$\frac{d\rho}{dt} = \frac{v\,[1 + if'(x)]}{\sqrt{1 + [f'(x)]^2}}$$

(2). If a point move with velocity v in the curve $r = f(\theta)$, represented in polar co-ordinates, prove that its velocity is

$$\frac{d\rho}{dt} = \frac{v\,[f'(\theta) + if(\theta)]}{\sqrt{[f'(\theta)]^2 + [f(\theta)]^2}} \operatorname{cis} \theta.$$

(3). If v = speed, and R = radius of curvature, prove that the tensor of acceleration is

$$\left| \frac{d^2\rho}{dt^2} \right| = \sqrt{\left\{\frac{dv}{dt}\right\}^2 + \frac{v^4}{R^2}}.$$

In the following curves let s = length of curve, θ = vectorial angle, x = abscissa in a Cartesian system. If in each a point move with speed v, determine the expressions for velocity, acceleration, and radius of curvature.

(4.) $\rho = \sqrt{a^2 + s^2} + ia \sinh \frac{s}{a}$.

$$\frac{d\rho}{dt} = \frac{v\,(s + ia)}{\sqrt{s^2 + a^2}}, \quad \frac{d^2\rho}{dt^2} = \frac{v^2\,(a^2 - ias)}{(s^2 + a^2)^{\frac{3}{2}}}.$$

(5) $\rho = ae^{\theta/n} \operatorname{cis} \theta$, the equiangular spiral.

$$\frac{d\rho}{dt} = \frac{v\,(1 + in)}{\sqrt{1 + n^2}} \operatorname{cis} \theta.$$

(6). $\rho a\theta \operatorname{cis} \theta$, the spiral of Archimedes.

(7). $\rho \frac{a}{\theta} \operatorname{cis} \theta$, the reciprocal spiral.

(8). $\rho = x + iae^{x/n}$, the logarithmic curve.

(9). $\rho = x + ic \cosh \frac{x}{c}$, the catenary.

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